

Seminar 4 in Applied Statistics

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💡 Problem 1

Consider the standard linear regression model with one regressor:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i.$$

a) Verify that the standard scalar OLS estimators of β_0 and β_1 coincide with

$$\hat{\beta} = (X^T X)^{-1} X^T Y,$$

where X is the $n \times 2$ design matrix whose i -th row is $(1, X_i)$.

b) Show that

$$P_X = X(X^T X)^{-1} X^T, \quad Q_X = I - P_X$$

are projection matrices.

💡 Problem 2

Consider the regression model

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 d_i + \beta_3 X_i d_i + \varepsilon_i,$$

where d_i is a dummy variable, i.e. $d_i \in \{0, 1\}$, and ε_i is the regression error.

Show that the OLS estimators of β_0 and β_1 in the full model coincide with the OLS estimators obtained by running the regression

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

only on the subsample for which $d_i = 0$.

💡 Problem 3

Suppose that (Y_i, X_i) , $i = 1, \dots, n$, are i.i.d. copies of random variables (Y, X) , and consider the linear regression model

$$Y_i = \alpha + \beta X_i + \varepsilon_i,$$

where ε_i is an unobserved regression error.

a) Find the probability limits of the OLS estimators $\hat{\alpha}$ and $\hat{\beta}$.

b) Under what conditions is the OLS estimator $\hat{\beta}$ consistent for β ?

💡 Problem 4 — FWL in Population Regression

Let $Y, X_1, \dots, X_p, \varepsilon$ be random variables, and define

$$X = (X_1, \dots, X_p)^T.$$

Assume that

$$\text{Cov}(X, \varepsilon) = 0_p.$$

Consider the linear model

$$Y = X^T \beta + \varepsilon.$$

Suppose that X can be partitioned into two groups:

$$X = \begin{pmatrix} D \\ W \end{pmatrix},$$

where D is a p_1 -dimensional vector of **target regressors** and W is a p_2 -dimensional vector of other regressors (control variables). Thus,

$$Y = D^T \beta_1 + W^T \beta_2 + \varepsilon.$$

The coefficient β_1 describes how the prediction of Y changes with D while holding W fixed.

Define the population partialling-out operator Q_W by

$$Q_W(V) = \tilde{V} = V - W^T \gamma_{VW},$$

where

$$\gamma_{VW} = \arg \min_{b \in \mathbb{R}^{p_2}} \mathbb{E} (V - W^T b)^2.$$

If V is a random vector, the operator is applied componentwise. Assume that all relevant random variables have finite second moments.

a) Show that Q_W is a linear operator.

b) Apply Q_W to both sides of

$$Y = D^T \beta_1 + W^T \beta_2 + \varepsilon.$$

Is the resulting regression error uncorrelated with the regressors? Express the population regression coefficient β_1 in terms of

$$\tilde{Y} = Q_W(Y), \quad \tilde{D} = Q_W(D).$$

c) How should the definition of Q_W be modified if the regression model also contains quadratic terms in W ?