

Homework 1 in Probabilistic Combinatorics

Your homework will be considered successfully defended if you can clearly explain the solutions of at least five of the following problems.

1. Prove that, if there is a real p , $0 \leq p \leq 1$, such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number $R(k, t)$ satisfies $R(k, t) > n$.

2. Using the result from Problem 1, show that

$$R(4, t) \geq \Omega\left(\frac{t^{3/2}}{(\ln t)^{3/2}}\right).$$

3. Suppose $n \geq 4$, and let H be an n -uniform hypergraph with at most $4^{n-1}/3^n$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors are represented.
4. Let $G = (V, E)$ be a graph with n vertices and minimum degree $\delta > 10$. Prove that there is a partition of V into two disjoint subsets A and B so that $|A| \leq O(n \ln \delta / \delta)$, and each vertex of B has at least one neighbor in A and at least one neighbor in B .
5. Let $G = (V, E)$ be a graph on $n \geq 10$ vertices, and suppose that if we add to G any edge not in G , then the number of copies of a complete graph on 10 vertices in it increases. Show that the number of edges of G is at least $8n - 36$.
6. Theorem 1.2.1 asserts that for every integer $k > 0$ there is a tournament $T_k = (V, E)$ with $|V| > k$ such that for every set U of at most k vertices of T_k there is a vertex v so that all directed arcs $\{(v, u) : u \in U\}$ are in E . Show that each such tournament contains at least $\Omega(k2^k)$ vertices.
7. We consider the random graph model $G(K_{n,n}, p)$, which is defined as follows. In the complete bipartite graph $K_{n,n}$, edges are independently deleted, each with probability $1 - p$.
Let $\lambda > 0$ and $p = 1 - \sqrt[n]{\lambda/n}$. What is the limit of the probability that in the random graph $G(K_{n,n}, p)$ there exists an isolated vertex in the upper part?
8. Find the expectation and variance of the number of cycles of length 4 in $G(n, n, p)$. For the variance, write the answer as a common fraction a/b for $n = 4$, $p = \frac{1}{2}$.