

List of Theorems

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1 Theorems Discussed in Class

Proposition 0.1 — Lower Bound for Ramsey Numbers

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then

$$R(k, k) > n.$$

Thus,

$$R(k, k) > \lfloor 2^{k/2} \rfloor$$

for all $k \geq 3$.

Theorem 0.2 — Property S_k

We say that a tournament T has **property** S_k if, for every set of k players, there exists a player who beats all of them.

If

$$\binom{n}{k} (1 - 2^{-k})^{n-k} < 1,$$

then there exists a tournament on n vertices that has property S_k .

Definition 0.3 — Dominating Set

A **dominating set** of an undirected graph $G = (V, E)$ is a set $U \subseteq V$ such that every vertex $v \in V \setminus U$ has at least one neighbor in U .

Theorem 0.4 — Upper Bound for the Domination Number

Let $G = (V, E)$ be a graph on n vertices with minimum degree $\delta > 1$.
Then G has a dominating set of at most

$$n \cdot \frac{1 + \ln(\delta + 1)}{\delta + 1}$$

vertices.

1.1 Hypergraph Two-Colorability

Let $H = (V, E)$ be an n -uniform hypergraph, meaning that each of its edges contains precisely n vertices.

We say that H has **property B**, or that it is **two-colorable**, if there exists a two-coloring of V such that no edge is monochromatic.

Let $m(n)$ denote the minimum possible number of edges of an n -uniform hypergraph that does not have property B.

Proposition 0.5 — Erdős (1963a)

Every n -uniform hypergraph with fewer than

$$2^{n-1}$$

edges has property B.

Therefore,

$$m(n) \geq 2^{n-1}.$$

Theorem 0.6 — Upper Bound for $m(n)$

$$m(n) < (1 + o(1)) \frac{e \ln 2}{4} n^2 2^n.$$

1.2 Intersecting Families

Definition 0.7 — Intersecting Family

A family $\mathcal{F}$ of sets is called **intersecting** if

$$A, B \in \mathcal{F} \implies A \cap B \neq \emptyset.$$

Suppose $n \geq 2k$, and let $\mathcal{F}$ be an intersecting family of k -element subsets of an n -set, for definiteness $\{0, \dots, n-1\}$.

Theorem 0.8 — Erdős–Ko–Rado

Let $n \geq 2k$, and let $\mathcal{F}$ be an intersecting family of k -element subsets of an n -element set. Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

1.3 Graph Theory and the Probabilistic Method

Theorem 0.9 — Erdős (1959)

For all integers k, ℓ , there exists a graph G such that

$$\text{girth}(G) > \ell \quad \text{and} \quad \chi(G) > k.$$

Here:

- $\text{girth}(G)$ denotes the length of the shortest cycle in G .
- $\chi(G)$ denotes the chromatic number of G .

Theorem 0.10 — Hamiltonian Paths in Tournaments

There exists a tournament T with n players and at least

$$n! 2^{-(n-1)}$$

Hamiltonian paths.

Theorem 0.11 — Bipartite Subgraphs

Let $G = (V, E)$ be a graph with n vertices and e edges.

Then G contains a bipartite subgraph with at least

$$\frac{e}{2}$$

edges.

Theorem 0.12 — Improved Bounds for Bipartite Subgraphs

Part (a). If G has $2n$ vertices and e edges, then it contains a bipartite subgraph with at least

$$\frac{en}{2n-1}$$

edges.

Part (b). If G has $2n+1$ vertices and e edges, then it contains a bipartite subgraph with at least

$$\frac{e(n+1)}{2n+1}$$

edges.

Theorem 0.13 — Lower Bound for the Independence Number

Let $G = (V, E)$ be a graph with n vertices and $\frac{nd}{2}$ edges, where $d \geq 1$.
Then

$$\alpha(G) \geq \frac{n}{2d},$$

where $\alpha(G)$ denotes the independence number of G .

2 Theorems Not Included in *The Probabilistic Method*

Theorem — Random Graphs: Independence and Chromatic Numbers

Let $G \sim G(n, \frac{1}{2})$.

Then, for every $\varepsilon > 0$, with high probability,

$$\frac{\chi_{\text{gr}}(G)}{\chi(G)} \leq 2 + \varepsilon,$$

and

$$\frac{\alpha(G)}{\alpha_{\text{gr}}(G)} \leq 2 + \varepsilon.$$

Here:

- $\chi(G)$ denotes the chromatic number.
- $\chi_{\text{gr}}(G)$ denotes the number of colors used by the greedy coloring algorithm.
- $\alpha(G)$ denotes the independence number.
- $\alpha_{\text{gr}}(G)$ denotes the size of an independent set produced by a greedy algorithm.

Remark. Within this course, only the inequality concerning the independence number will be proved. The proof for the chromatic number proceeds analogously.