Problem set 6 - CLT & WLLN

During the seminars, we will work on problems across three different categories of difficulty, with Category 1 being the easiest and Category 3 the most challenging.

CATEGORY 1:

Weak Law of Large Numbers (WLLN). Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with finite mean μ. Then

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu$$

as $n \to \infty$.

Central Limit Theorem (CLT). Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with finite mean μ and finite variance $\sigma^2 > 0$. Then

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$

as $n \to \infty$.

Poisson Limit Theorem. Let $X_n \sim \text{Binomial}(n, p_n)$ where $p_n \to 0$ and $np_n \to \lambda > 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \Pr\bigl(X_n = k\bigr) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

i.e., X_n converges in distribution to a Poisson random variable with parameter λ.

1. 1800 dice are thrown. Find an approximate value for the probability that the total number of occurrences of 2 and 6 is not less than 620.
2. When typing, the stenographer makes a mistake in the character with probability of 0.0005. Find an approximate value for the probability that, when typing 10.000 characters, the stenographer will make a mistake no more than three times.
3. A bank teller serves customers standing in the queue one by one. Suppose that the service time ξ_i for customer i has mean $\mathsf{E}\xi_i = 2$ (minutes) and $\mathsf{Var}(\xi_i) = 1$. We assume that service times for different bank customers are independent. Let η be the total time the bank teller spends serving 50 costumers. Find $\mathsf{P}(90 < \eta < 110)$.
4. The probability that a light bulb lasts more than 1000 hours is $\frac{1}{3}$. Estimate the probability that out of 1800 light bulbs, the lifetime of at least 580 bulbs exceeds 1000 hours.

CATEGORY 2:

5. A die is thrown n times. Let the random variable S_n correspond to the sum of the points obtained. Find n such that $\mathsf{P}(\bigl|\frac{S_n}{n} - 3.5\bigr| \geq 0.1) \geq 0.05$

   a) Using the Chebyshev's inequality.

   b) Using the Central Limit Theorem.

6. How many times should we throw a die if we want that the sum of points obtained was at least 4500 with probability $p \geq 0,975$? (use the central limit theorem).

CATEGORY 3:

7. Prove that

$$\lim_{n\to\infty} \frac{1}{e^n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2},$$

using the Central Limit Theorem.

8. Let $\{\xi_n\}$ be a sequence of independent identically distributed random variables with distribution $\mathcal{U}(0,1)$. Set $M_n = \max\{\xi_1, \ldots, \xi_n\}$. Prove that

$$n(1 - M_n) \xrightarrow{\mathsf{d}} \mathsf{Exp}(1)$$

9. Let $\{\xi_n,\, n \in \mathcal{N}\}$ be independent random variables with finite variance. Prove that for any $x \in \mathbb{R}$ the following limit exists:

$$\lim_{n\to\infty} P(\xi_1 + \ldots + \xi_n \le x)$$

and that it equals 0, 1, or 1/2. Specify the conditions under which each of these values occurs.