Problem set 2 - Random variables and random vectors

1. Let ξ be a random variable with the following PDF

$$p_\xi(x) = \begin{cases} 0 & , x < 1; \\ \frac{3}{x^4} & , x \geq 1. \end{cases}$$

Let $\eta = \frac{1}{\xi}$. Find the CDF and PDF of η.

2. The PDF of the random variable X is given by $p_X(x) = cx^{-3/2} I(x \geq 1)$.
   (a) Find c, $F_X(x)$.
   (b) Let $Y = \frac{1}{X}$. Find $p_Y(y)$ and $\mathsf{P}(0,1 < Y < 0,2)$.

3. The joint PDF of X and Y is given by

$$p_{X,Y}(x,y) = \begin{cases} c(x^2 + y^2) & , 0 \leq x \leq 2, 0 \leq y \leq 2, ; \\ 0 & , \text{otherwise} \end{cases}$$

Find c, $p_Y(y)$, $\mathsf{P}(Y > 1)$.

4. The joint PDF of X and Y is given by

$$p_{X,Y}(x,y) = \begin{cases} 2\lambda x e^{-\lambda y} & , 0 \leq x \leq 1, y \geq 0; \\ 0 & , \text{в остальных случаях.} \end{cases}$$

Are X and Y independent random variables?

5. Random variable ξ takes values in the interval (a, b), where $-\infty \leq a < b \leq \infty$, and has density $f(x)$. The function $\varphi(x)$ is strictly monotone and differentiable on (a, b). Moreover $\varphi'(x) \neq 0$ in (a, b). Find the density of the random variable $\eta = \varphi(\xi)$. Find the PDF of $\sqrt{\xi}$, if ξ has an exponential distribution with parameter λ.

6. Let ξ_1, ξ_2 be independent random variables, uniformly distributed on $[0, 2]$. Find $\mathsf{P}(1 \leq \xi_2\xi_2 \leq 2)$.

7. Let $\xi \sim \mathsf{Exp}(\lambda)$. Find the PDF of the following random variables
   (a) ξ^k, $k \in \mathrm{N}$
   (b) $\frac{1}{\lambda} \ln \xi$
   (c) $\{\xi\}$, where $\{\cdot\}$ is the fractional part function
   (d) $1 - e^{-\alpha\xi}$.

8. The PDF of a random vector (ξ, η) is given by $p_{(\xi,\eta)}(x,y) = \frac{1}{\pi/4} I(x^2 + y^2 < 1, x > 0, y > 0)$. Find the PDF of the random variable $\xi + \eta$.

9. Let $\xi_1, \ldots, \xi_n$ be i.i.d. random variables with CDF F_ξ. Let $\xi_{(1)}, \ldots, \xi_{(n)}$ be the same sequence but ordered in increasing order (i.e. $\xi_{(1)} \leq \ldots \leq \xi_{(n)}$). Find
   (a) the CDF of $\xi_{(k)}$
   (b) the PDF of $\xi_{(k)}$