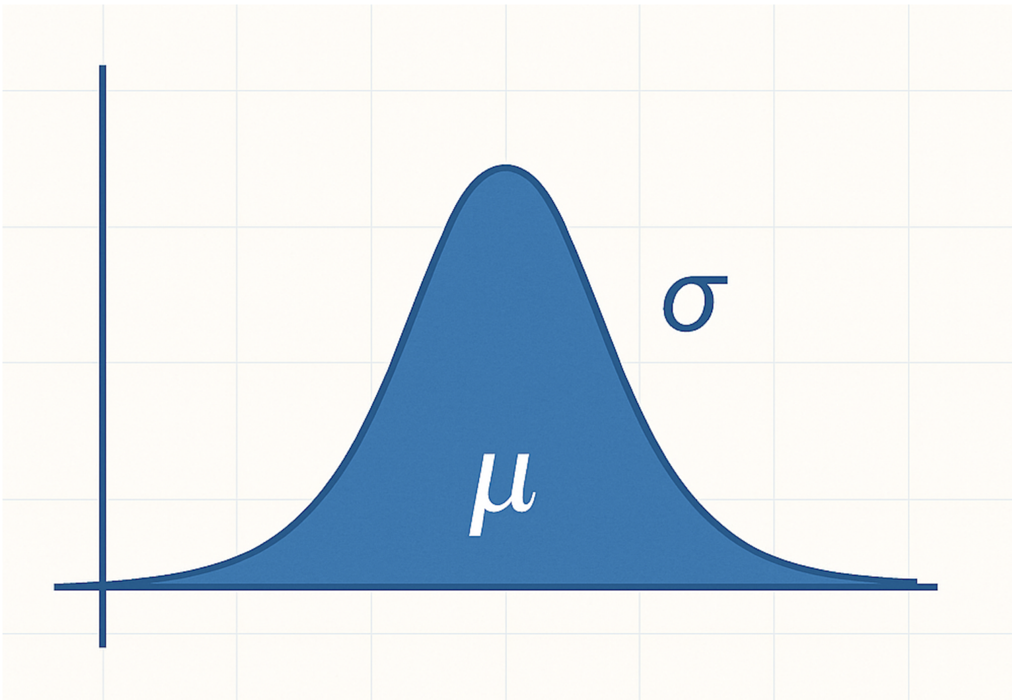
Higher School of Economics

Probability Theory and Statistics: Problem Sets for ***Module 1***

Topics:

Seminar 1:	*Combinatorics and Classical probability.*
Seminar 2:	*Geometric probability. Bernoulli scheme.*
Seminar 3:	*Conditional Probability. Bayes Theorem.*
Seminar 4:	*Discrete Random Variables and Discrete Joint Distributions.*
Seminar 5:	*Discrete Expectation and Variance.*
Seminar 6:	*Discrete Covariance and correlation.*
Seminar 7:	*Continuous Random Variables. Normal Distribution.*
Seminar 8:	*Review before Midterm.*



DSBA

Fall 2025

Basics of probability. Combinatorics

During the seminars, we will work on problems across three different categories of difficulty, with Category 1 being the easiest and Category 3 the most challenging.

CATEGORY 1:

1. Suppose that a class of 100 students consists of four subgroups, in the following proportion:

	Men	Women
Taking Economics	17%	38%
Not taking Economics	23%	22%

What is the chance that a randomly chosen student is:

(a) a woman?
(b) taking economics?
(c) a man or taking economics?
(d) a woman and taking economics?

2. Suppose that in families with three children births are independent, and the probability of a boy on each birth is 52 %. Use the table

Outcome	BBB	BBG	BGB	BGG	GBB	GBG	GGB	GGG
Probability	0.14	0.13	0.13	0.12	0.13	0.12	0.12	0.11

and find the chance that in a family of three children, there will be:

(a) exactly 2 girls;
(b) at least two girls;
(c) at least one child of each sex;
(d) the middle child being opposite in sex to the other two.

3. You take 3 cards from a deck of 36 cards. After you take each card you return it to the deck and shuffle the deck.

(a) What is the set of elementary outcomes and what are their probabilities?
(b) What is the probability to get (queen, king, ace)? (The first card is a queen, the second card is a king, the third card is an ace).
(c) What is the probability to get (king, king, ace)?

4. You take 3 cards from a deck of 36 cards. After you take each card you do not return it to the deck. Answer (a), (b), (c) from the previous problem.

5. Find unions and intersections of the following events. In which case one event is a subset of the other?

(a) $A = \{1, 2, 5, 6\}, B = \{1, 5\}$.
(b) $A = \{Ann, Mary, Mike\}, B = \{Tom, Mike, John\}$.
(c) $A = \{Moscow, London, Paris\}, B = \{Paris, Berlin, Tokyo\}$, $C = \{Tokyo, Rome\}$.

6. (a) If A and B are mutually exclusive events with probabilities of 0.6 and 0.2 respectively, then what is the probability of A or B occurring?
(b) If $(A) = 0.2$, $(B) = 0.3$, and $(A \cap B) = 0.1$, then what is $(A \cup B)$?
(c) If $(A) = 0.4$, $(B) = 0.5$, and $(A \cup B) = 0.7$, then what is $(\overline{A \cap B})$?

7. A survey of the houses in an old residential area found 30% with holes in the roof, 40% with broken windows, and 25% with the both problems.

(a) What is the proportion of houses with one or the other (or both) problems?
(b) What is the proportion of houses with holes in the roof but without broken windows?
(c) What is the proportion of houses with exactly one of these problems?
(d) What is the proportion of houses with none of these problems?

8. Suppose a word is picked at random from this sentence.
   (a) What is the sample space of this random experiment?
   (b) Find the probability that:
      i. the word has at least 4 letters;
      ii. the word contains at least 2 vowels;
      iii. the word contains at least 4 letters and at least 2 vowels.

9. In a group of students (none of whom are from DSBA) 25% smoke hookah, 60% drink alcohol, and 15% do both. What fraction of students have at least one of these bad habits?

10. 20 families live in a neighborhood of Wolfenstein Castle: 4 have 1 child, 8 have 2 children, 5 have 3 children, and 3 have 4 children. If we meet a local child near the castle, what are the probabilities p_1, p_2, p_3, p_4, that the child comes from a family with 1, 2, 3, 4 children?

11. Suppose we roll a red die and a green die. What is the probability that the number on the red die is larger than the number on the green die?

12. Two dice are rolled. Find the probability that
    (a) the two numbers will differ by 1 or less;
    (b) the maximum of the two numbers will be 5 or larger.

13. In Galileo's time people thought that when three dice were rolled, a sum of 9 points and a sum of 10 points had the same probability, since each could be obtained in 6 ways:

$$\begin{array}{rllllll} 9: & 1+2+6, & 1+3+5, & 1+4+4, & 2+2+5, & 2+3+4, & 3+3+3. \\ 10: & 1+3+6, & 1+4+5, & 2+4+4, & 2+3+5, & 2+2+6, & 3+3+4. \end{array}$$

    (a) Compute the probability of the event 1 + 2 + 6 (one point on one die, two points on the other, and 6 points on the remaining die).
    (b) Compute the probability of the event 2 + 4 + 4 (two points on one die, and four points on each of the remaining dice).
    (c) Find the probabilities of the events A = "A total of 9 points on three dice", and B ="A total of 10 points on three dice".

CATEGORY 2:

14. Suppose n friends are sitting around a table in random order. What is the probability that A sits next to B?

15. In a box there are 10 white balls and 6 black. 4 balls are chosen randomly from among them without return. What is the probability that, among the chosen balls
    (a) there is at least one black ball
    (b) there are exactly two black balls

16. A closet contains n pairs of boots. $2r$ boots are randomly selected from the closet. What is the probability that among the selected boots
    (a) There are not complete pairs.
    (b) There is exactly one complete pair.

17. Find the probability that among 50 students attending a lecture on probability theory, at least two of them have the same date of birth.

18. In a box there are 28 black balls and 4 white. 10 balls are chosen randomly from among them. What is the probability that, among the chosen balls

    (a) there is at least one white ball
    (b) there is exactly one white balls
    (c) there are at least two white balls
    (d) there are exactly two white balls

19. A deck of playing cards contains 52 cards, divided into 4 different suits of 13 cards each. 6 cards are randomly drawn. Find the probability that

    (a) Among these cards there will be the king of spades.
    (b) Among these cards there will be a representative of each suit.

CATEGORY 3:

20. Two M-sided dice are thrown. Find the probability that the sum of the two numbers obtained is equal to i.

21. A set of n balls is randomly placed into m boxes. Find the probability that all boxes are non-empty if the balls are distinguishable.

22. A shelf holds 12 books in a row. We pick 5 books randomly. Find the probability that no pair of adjacent books is chosen.

23. Some residents of Dolgoprudny consider a tram ticket "special" if the sum of the first three digits, of its six-digit number, is equal to the sum of the last three digits. Find the probability of getting a "lucky" ticket.

Geometric probability. Bernoulli scheme

During the seminars, we will work on problems across three different categories of difficulty, with Category 1 being the easiest and Category 3 the most challenging.

CATEGORY 1:

1. Two points X and Y are randomly chosen on an interval $OA = [0, 1]$. Find the probability of each of the following events:
   (a) A distance between X and O is less than $\frac{1}{10}$.
   (b) A distance between X and O is between 0.7 and 0.705.
   (c) A distance between X and O is equal to 0.7.
   (d) A distance between X and Y is less than 0.5.
   (e) A distance between X and Y is equal to $\frac{1}{3}$.
   (f) Length of XY is less than the distance between O and the closest point to it.
2. Consider a round shooting mark of radius R. Someone shoots to it with bullets of radius B. Find the probability that a hole made in the shot entirely lies in a interior circle of radius r. Assume $R > r > B$.
3. You have a biased coin for which $(H) = p$. You toss the coin 20 times. What is the probability that:
   (a) you observe first 8 heads and then 12 tails,
   (b) you observe 8 heads and 12 tails,
   (c) you observe more than 8 heads and more than 8 tails?
4. Suppose a good password must consist of two lowercase letters (a to z), followed by one capital letter (A to Z), followed by four digits. For example, *"ejT3018"* is a good password.
   (a) Find the total number of good passwords.
   (b) A hacker wrote a program that randomly generated 10^8 good passwords (one password could be generated more than once). What is the probability that at least one of the generated passwords matches the password of a particular user?
   (c) Answer the question b) assuming that the program generated 10^8 distinct passwords.
5. The student has learned 20 out of 25 exam questions before the exam. She will be asked 3 different questions. If she answers all the questions, she will receive an excellent mark; if she answers 2 questions, she will receive a good mark; and if she answers 0 or 1 question, she will receive an unsatisfactory mark. Find the probability of the following:
   (a) obtaining an excellent mark,
   (b) receiving an unsatisfactory mark,
   (c) passing the exam,
   (d) passing the exam if she knows the answer to the first question,
6. From a brood of mice, containing two white specimens, four mice are taken at random (without return). The probability that both white mice were taken is twice as likely as the probability that neither was taken. How many mice are there in the brood?

CATEGORY 2:

7. Find the probability that in $2n$ trials of the Bernoulli scheme with probability of success p and failure $q = 1 - p$ there will be $m + n$ successes and all trials with even numbers will end in success.

8. A rod of lenght l is broken at two randomly chosen points. Find the probability that the obtained segments can form a triangle.

9. Consider two coins. Let $A = \{\text{first coin lands heats}\}$, $B = \{\text{second coin lands heats}\}$,

   $C = \{\text{heats appeared only once}\}$. Are A, B and C mutually independent?

10. Three dice are thrown. Events A, B and C stand for rolling matching numbers (two 6s, for example) on the first and second, on the second and third, on the first and third dice, respectively. Are these events pairwise and mutually independent?

11. We randomly order the numbers from 0 to 99, getting the sequence $(x_1, x_2, \ldots, x_{100})$. Are the following events independent: «$x_{80} > x_{81}$» and «$x_{81} > x_{82}$».

12. A particle changes its position by one unit every second. It moves to the right with probability p and to the left with probability $1 - p$. Let x_t correspond to the position of the particle at time t. Find the probability that at time $t = n$ the particle moved k units to the right. Let $x_0 = 0$.

CATEGORY 3:

13. A point A is randomly picked from the inside of a rectangle of sides 1 and 2. Find the probability of the following events:

    (a) The distance from A to the nearest diagonal is at most x.

    (b) The distance from A to each side is at most x.

14. Find the probability that three randomly chosen segments of length no more than 1 can form a triangle.

15. (*Bertrand paradox*) Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle? Consider 3 different methods: the "random endpoints"method, the "random radial point"method and the "random midpoint"method.

16. In the Bernoulli scheme, the probability of success (getting 1) is $p < 1$. Find the probability that in an infinite sequence of trials:

    (a) 00 will appear before 01;

    (b) 00 will appear before 10.

17. There are a white and b black balls in a box. Someone takes balls from the box, one at a time. A_k is the event that, at time k, he picks a white ball. Are the events $A_1, \ldots, A_n$ independent?

18. Consider the Bernoulli scheme. Let A_i be a random subset of $\{1, 2, \ldots, n\}$. Consider m such sets. Find **a)** $\mathsf{P}(A_i \cap A_j = \emptyset)$ for arbitrary i, j. **b)** $\mathsf{P}(A_1 \subset A_2 \cap A_3)$.

19. Consider the Bernoulli scheme. Let A_i be a random subset of $\{1, 2, \ldots, n\}$. Consider m such sets. Find $\mathsf{P}(|A_1 \cup \cdots \cup A_m| = k)$.

Conditional Probability. Bayes' Theorem

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CATEGORY 1:

1. There are three cards. The letter A is written on both sides of the 1st card; the letter A is written on both sides of the 2nd card; letters A and B are written on different sides of the 3rd card.
A random card has been put on the table, such that the letter A can be seen.
What is the probability that the letter A is written on the other side of the card?

2. A system consists of two parallel elements and is working if at least one of them is working.
At a random time the 1st element is out of order with probability 0.1, the 2nd is out of order with probability 0.2.
Someone told us that now the system is working. What is the probability that the 2nd element is out of order?

3. Rachel is going to a party. There is a 60% chance Ross will go too. If Ross does not go, there is a 20% chance she will enjoy herself. If Ross does go, there is a 70% chance she will enjoy herself.
   a) What is the probability that Rachel will enjoy the party?
   b) Suppose you know Rachel did not enjoy herself. What is the probability that Ross was not present?

4. A student can enter a course either as a beginner (73% of all students) or as a transferring student (27% of all students). It is found that 62% of beginners eventually graduate, and that 78% of transferring students eventually graduate. Find:
   a) the probability that a randomly chosen student is a beginner who will eventually graduate
   b) the probability that a randomly chosen student will eventually graduate
   c) the probability that a randomly chosen student is either a beginner or will eventually graduate, or both
   d) Are the events 'Eventually graduates' and 'Enters as a transferring student' statistically independent?
   e) If a student eventually graduates, what is the probability that the student entered as a transferring student?
   f) If two entering students are chosen at random, what is the probability that not only do they enter in the same way but that they also both graduate or both fail?

5. Two snipers shoot a target. Sniper A hits with probability 0.5 and sniper B hits with probability 0.8. They toss a fair coin to determine who shoots first. Find the probability that sniper A shot first if it is known that after the first shot the target was hit.

6. A player picks a spot at random within a region S on a flat surface. S is split into four sections, each covering 50%, 30%, 12%, and 8% of the total S area. If the chosen spot falls into one of these sections, the player wins a prize with probabilities of 0.01, 0.05, 0.20, and 0.50, respectively.
The player has now selected a spot and won a prize. Which section of the S area is the most likely location for the chosen spot?

CATEGORY 2:

7. The Department of Public Health is giving a free medical test for a certain disease. The test is 90% reliable in the following sense: if a person has the disease, there is a probability of 0.9 that the test will give a positive response; whereas, if a person does not have the disease, there is a probability of only 0.1 that the test will give a positive response. Data indicate that your chances of having the disease are only 1 in 10,000. You decide to take the test. A few days later you learn that you had a positive response to the test.
What is the probability that you have the disease?

8. Consider a high-risk population where 5% of people have COVID-19. A diagnostic test is correct in 95% of cases if a person has COVID-19 and in 90% of cases if a person does not have COVID-19. If a person tests positive (indicating COVID-19), what is the probability that the person does **not** have COVID-19?

9. Three dice are rolled. Find the probability of getting 6 on at least one die if

   a) The numbers shown on the three dice are different.

   b) The number shown on the first and on the third die is the same.

10. In box A there are 10 white balls and 20 black balls. In box B there are 10 white balls and 10 black balls. We randomly pick 4 balls out of box A and 6 balls out of box B and put them all together in an empty box C. What is the probability of picking a white ball from box C?

11. The exam in probability theory consists of n questions. k of these questions are "easy". Find the probability that in a group of n students **a)** The first student chooses an easy question. **b)** The ith student chooses an easy question.

12. There are 10 phones in a store, for which the probability of proper functioning within a month is 0.9 and 5 phones with a similar probability of 0.95. Find the probability that two phones, bought randomly from the store, will work properly for a month.

13. In a box there are N balls. M of them are white. We take out n balls and put them back. Consider the following events: $A_k = \{\text{in the k-th trial we took a white ball}\}$,

    $B_m = \{\text{we took out } m \text{ white balls in total}\}$. Find $\mathsf{P}(A_k|B_m)$.

CATEGORY 3:

14. All tickets (100 tickets) for the Moscow-Berlin flight have been purchased. All passengers arrived for boarding. The first person to enter the plane is an old woman who sits in a random seat. Passengers enter in turn, and each next passenger takes his seat, if it is free. If the seat is occupied, then the passenger sits in a random seat from the remaining ones. What is the probability that the last passenger will take their seat?

15. *(Monty Hall problem)* Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?"Is it to your advantage to switch your choice?

16. Two players conduct an endless series of independent tests. In each test, player A rolls 3 dice and player B rolls 2 dice at the same time. They carry out these tests until they obtain "six"on at least one of the dice. Find the probability of the following events:

    (a) The first "six" was rolled by player A, and not by B.

17. From an urn containing a white and b black balls, two players take turns drawing balls. The winner is whoever chooses the white ball first. Find the probability that the first player wins in cases where the balls are drawn

    (a) under the scheme of equiprobable choice with return;

    (b) under the scheme of equiprobable choice without replacement.

Discrete Random Variables. Joint Discrete Distributions

During the seminars, we will work on problems across three different categories of difficulty, with Category 1 being the easiest and Category 3 the most challenging.

CATEGORY 1:

1. A box contains two gold balls and three silver balls. You are allowed to choose successively balls from box at random. You win 1 dollar each time you draw a gold ball and lose 1 dollar each time you draw a silver ball. After a draw, the ball is not replaced. Show that, if you draw until you are ahead by 1 dollar or until there are no more gold balls, this is a favorable game.

2. Sketch the c.d.f. for a random variable X:

x	0	1	2	4
$P_X(x)$	$1/3$	$1/3$	$1/6$	$1/6$

3. Consider two random variables with the following joint distribution

$X\backslash Y$	1	2
3	$1/4$	$1/4$
5	$1/6$	$1/3$

1) Find the marginal distributions of X and of Y.
2) Are X and Y independent?

4. A coin is tossed 3 times. Let random variable ξ denote the number of tails obtained. Let η denote the winnings earned in a single play of a game with the following rules:

-Player wins \$1 if first tails occurs on the first toss,

-Player wins \$2 if first tails occurs on the second toss,

-Player wins \$3 if first tails occurs on the third toss,

-Player loses \$1 if no tails occur.

Find the Joint PMF of ξ and η.

5. There are 100 cards in a box, labeled with numbers $1, 2, \ldots, 100$. A card is drawn at random from the box, and it is immediately returned to the box. We repeat this procedure 200 times. Find the approximate value of the probability that the card with the number 1 will appear exactly 3 times.

6. There is a set of four cards marked the numbers $-1, 0, 1, 2$. The player draws one card at random, writes down the number with which this card is marked, and returns the card back. Then he repeats the above procedure again. Let X_i be the number written during the i-th experiment ($i = 1, 2$). Find the PMF of the random variables $Y = X_1X_2$ and $Z = X_1 + X_2$.

7. The Joint PMF of the random vector (X, Y) is given by the following table:

X$\backslash Y$	-1	0	1
-2	$1/8$	$1/4$	$1/8$
2	$1/12$	$1/3$	$1/12$

(a) Find the PMF of X and $\mathsf{P}(Y \geq 0)$.

(b) Find the PMF of $X + Y$.

(c) Find the PMF of $Z = \min\{X, Y\}$.

8. There is a set of four cards marked the numbers $-1, 0, 1, 2$. The player draws one card at random, writes down the number with which this card is marked, and does not return the card back. Then he repeats the above procedure again. Let X_i be the number written during the i-th experiment ($i = 1, 2$). Find the PMF of the random variables $Y = X_1X_2$ and $Z = X_1 + X_2$.

9. You roll one red die and one green die. Define the variables ξ and η as follows:

   ξ = The number showing on the red die; η = The number of dice that show the number six. For example, if the red and green dice show the numbers 6 and 4, then $\xi = 6$ and $\eta = 1$. Write down the table showing the joint probability mass function for ξ and η, find the marginal distribution for η, and compute $\mathsf{E}\eta$.

10. The Joint PMF of the random vector (X, Y) is given by the following equality

$$\mathsf{P}(X = k, Y = j) = p^k(1-p)\frac{\lambda^j}{j!}e^{-j}, \quad k = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots;$$

   where $0 < p < 1$, and $\lambda > 0$. Find the marginal PMF of X and Y.

CATEGORY 2:

11. The PMF of ξ is given by $\mathsf{P}(\xi = k) = \frac{c}{k(k+1)(k+2)}$, $k = 1, 2, \ldots$. Find **a)** The value of c. **b)** $\mathsf{P}(\xi \geq 3)$.

12. Let ξ, η be two independent random variables such that $\xi \sim Bin(n, p)$ and $\eta \sim \mathrm{Bin}(m, p)$. Find the probability distribution of $\xi + \eta$.

13. Let P be a discrete distribution on $\mathbb{R}$, $p(x) = \mathsf{P}(\{x\})$.

    (a) $p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x \in \mathbb{Z}_+$ (Poisson distribution). Find $\mathsf{P}(2\mathbb{Z}_+)$.

14. A die is rolled k times. Let X_i, $i = 1, 2, \ldots, k$, be the number that appeared on the upper face of the die during the i-th flip.

    (a) Find the PMF of $Y_k = \max\{X_1, \ldots, X_k\}$.

    (b) Find the PMF of $Z_k = \min\{X_1, \ldots, X_k\}$.

    (c) Find the PMF of $Z_k = \max\{4, X_1\}$.

15. Let ξ, η be two independent random variables such that $\xi \sim Poiss(\lambda_1)$ and $\eta \sim \mathrm{Poiss}(\lambda_2)$. Find the probability distribution of $\xi + \eta$.

CATEGORY 3:

16. Consider the following sequence of independent random variables $\{\xi_i\}_{i=1}^n$, such that $\xi_i \sim \mathsf{Geom}(p_i)$. Prove that

$$\min\{\xi_1, \ldots, \xi_n\} \sim \mathsf{Geom}\left(1 - \prod_{i=1}^n (1 - p_i)\right)$$

17. Let ξ, η be two independent random variables such that $\xi \sim Poiss(\lambda_1)$ and $\eta \sim \mathrm{Poiss}(\lambda_2)$. Find $\mathsf{P}\{\xi = k | \xi + \eta = n\}$.

18. Let P be a discrete distribution on $\mathbb{R}$, $p(x) = \mathsf{P}(\{x\})$.

    (a) $p(x) = p(1-p)^{x-1}$, $x \in \mathbb{N}$ (geometric distribution). Find its CDF and $\mathsf{P}(2\mathbb{Z}_+)$.

Discrete Expectation and Variance

During the seminars, we will work on problems across three different categories of difficulty, with Category 1 being the easiest and Category 3 the most challenging.

CATEGORY 1:

1. A number is chosen at random from the set $S = \{-1, 0, 1\}$. Let X be the number chosen. Find the expected value, variance, and standard deviation of X.

2. The random variable X takes the values 0, 1, and 4 according to the following probability distribution:

x	0	1	4
$P_X(x)$	0.2	k	k

   a) Determine the constant k.
   b) Find $\mathbb{E}(X)$.
   c) Find $\mathrm{Var}(X)$.

3. A random variable X has the following distribution:

x	0	1	2	4
$P_X(x)$	1/3	1/3	1/6	1/6

   Find $\mathbb{E}(X)$, $\mathbb{E}(X(X+1))$, $\mathrm{Var}(X)$, and $\sigma(X)$.

4. X is a random variable with $\mathbb{E}(X) = 100$ and $\mathrm{Var}(X) = 15$. Find:
   a) $\mathbb{E}(X^2)$
   b) $\mathbb{E}(3X + 10)$
   c) $\mathbb{E}(-X)$
   d) $\mathrm{Var}(-X)$
   e) $\sigma(-X)$

5. A fair coin is tossed three times. Let X be the number of heads that turn up. Find $\mathrm{Var}(X)$ and $\sigma(X)$.

6. Let ξ_1, ξ_2 represent the values obtained in two different dice. Find $\mathsf{E}(\xi_1)$, $\mathsf{E}(\xi_1 + \xi_2)$, $\mathsf{E}(\xi_1\xi_2)$.

CATEGORY 2:

7. A random variable X has a binomial distribution with mean 10 and variance 6. Find $P(X = 4)$.

8. A box contains 10 white balls and 2 black balls. 6 balls are selected at random. Random variable X is equal to the number of black balls in the 6 selected.
   a) Find the distribution of random variable X.
   b) Find expected value $\mathbb{E}(X)$.
   c) Find expected value of X, given that $X > 0$, i.e. $\mathbb{E}(X \mid X > 0)$.

9. Find the expected value and the variance of a random variable such that
   (a) $\xi \sim \mathsf{Ber}(p)$

(b) $\xi \sim \mathsf{Bin}(n, p)$

(c) $\xi \sim \mathsf{Pois}(\lambda)$

(d) $\xi \sim U\{1, \ldots, N\}$.

(e) $\xi \sim Geom(p)$

10. A box contains m white balls and n black balls. We randomly pick a ball (and then we return it) until we obtain a white ball. Find the Expected value and the Variance of the number of balls taken out.

11. A group of n students throw their hats at their graduation. After that, each of them picks up from the floor a hat at random. Find the expected value and the variance of the number of students who get the correct hat.

12. Let $\xi \sim \mathsf{Pois}(\lambda)$. Find $\mathsf{E}e^{\xi}$.

13. A basket contains 4 apples and 6 oranges. We randomly choose from it 3 fruits. Let ξ denote the number of apples among the selected fruits. Find $\mathsf{E}\xi$, $\mathsf{Var}\xi$, $\mathsf{P}(\xi > 1)$.

14. Suppose we can roll a fair N-sided die up to m times (each face of the die contains an exclusive number of points, from 1 to N). Let ξ correspond to the number of times that face showing 1 point appears. In the same way, let η represent the sum of all the points obtained during the game.

    Find $\mathsf{E}\xi, \mathsf{Var}\xi$ and $\mathsf{E}\eta, \mathsf{Var}\eta$

CATEGORY 3:

15. For a natural number k, the k-th factorial moment of a random variable ξ is $\mathsf{E}[\xi(\xi-1)(\xi-2)\ldots(\xi-k+1)]$. Prove that for $\xi \sim \mathsf{Pois}(\lambda)$, its k-th factorial moment is λ^k.

16. A coin is tossed 1000 times. Making use of the Chebyshev's Inequality, estimate the probability that the number of times we obtained Tails will be in the interval $[450, 550]$.

17. A person who has n keys wants to unlock his door by testing the keys independently and in random order. We know that keys that do not match are not excluded from further tests. On the other hand, it is known that only one of the available keys fits the lock. Find the expectation and variance of the number of trials.

    (a) Solve the problem provided that keys that don't fit are excluded from further tests.

Discrete Covariance and Correlation

During the seminars, we will work on problems across three different categories of difficulty, with Category 1 being the easiest and Category 3 the most challenging.

CATEGORY 1:

1. Suppose that X and Y have the following joint probability mass function:

$X \backslash Y$	1	2	3
1	0.25	0.25	0
2	0	0.25	0.25

What is the correlation coefficient?

2. The probability distribution of a random variable X is:

$X \backslash Y$	-1	0	1
$P_X(x)$	a	b	a

What is the correlation coefficient between X and X^2?

CATEGORY 2:

3. A box contains 100 balls, where 25 of them are white. Two balls are successively taken out of the box. Let ξ_i be the number of white balls appearing in the i-th removal ($i = 1, 2$). Find the correlation coefficient between ξ_1 and ξ_2.

4. Two dice are rolled. Let ξ_1 correspond to the value obtained in the first die, and let ξ_2 correspond to the value obtained in the second die. On the other hand, let $\eta_1 = \xi_1 + \xi_2$ and $\eta_2 = \xi_1 - \xi_2$. Find $\mathsf{cov}(\eta_1, \eta_2)$. Are η_1 and η_2 independent?

5. Consider two random variables X and Y. They both take the values 0, 1, and 2. Joint probabilities for each pair are given by the following:

$Y \backslash X$	0	1	2
0	0	0.2	0.2
1	0.2	0	0.1
2	0.2	0.1	0

a) Calculate marginal distributions, expected values, and covariance of X and Y.

b) Calculate covariance of the random variables X and V, where $V = X - Y$.

c) Calculate $\mathbb{E}(X \mid Y = 0)$ and $\mathbb{E}(X \mid V = 1)$.

d) The random variable W has the same marginal distribution as X and the random variable Z has the same distribution as Y. It is also known that W and Z are independent. Write down the table for the joint probabilities of W and Z.

CATEGORY 3:

6. Let ξ and η be two random variables such that $\mathsf{P}(\xi\eta = 0) = 1; \mathsf{P}(\xi = 1) = \mathsf{P}(\xi = -1) = \mathsf{P}(\eta = 1) = \mathsf{P}(\eta = -1) = \frac{1}{4}$. Find the joint probability distribution of these two random variables and $\mathsf{E}\xi$, $\mathsf{E}\eta$, $\mathsf{Var}\xi$, $\mathsf{Var}\eta$, $\mathsf{cov}(\xi, \eta)$.

7. A die is rolled n times. Let ξ and η correspond to the number of times we obtained 1 and 6, respectively. Find the correlation coefficient of these two random variables.

Continuous Random Variables. Normal Distribution.

During the seminars, we will work on problems across three different categories of difficulty, with Category 1 being the easiest and Category 3 the most challenging.

CATEGORY 1:

1. Let X be a random variable with uniform distribution on the interval $(-2, 3)$. Find $\mathbb{P}(X > 2 \mid X > 1)$.
2. Consider a normally distributed random variable $Z \sim N(0, 1)$.
   a) What is $P(Z > 1.2)$?
   b) What is $P(-1.24 \leq Z \leq 1.86)$?
   c) The probability is 0.70 that Z is less than what number?
   d) The probability is 0.60 that Z is greater than what number?
3. Consider a normally distributed random variable $X \sim N(5, 4)$.
   a) What is $P(X > 6.4)$?
   b) What is $P(5.8 \leq X < 7.0)$?
   c) The probability is 0.05 that X is in the symmetric interval about the mean between which two numbers?
4. Consider a normally distributed random variable $X \sim N(6, 25)$. Find the following probabilities:
   a) $P(6 < X < 6.4)$
   b) $P(0 \leq X < 8)$
   c) $P(-2 < X \leq 0)$
5. The manufacturer of a brand new lithium battery claims that the mean life of a battery is 3800 hours with a standard deviation of 250 hours.
   a) What percentage of batteries will last for more than 3500 hours?
   b) What percentage of batteries will last for more than 4000 hours?
   c) Batteries which will last for more than c hours constitute more than $\frac{1}{5}$ of the population. Find the maximum possible c.
6. Consider a normally distributed random variable $X \sim N(\mu, \sigma^2)$. It is known that $P(X \leq 66) = 0.0359$ and $P(X \geq 66) = 0.1151$.
   a) Give a clearly-labelled sketch to represent these probabilities on a normal curve.
   b) Show that the value of σ is 5.
   c) Find $P(69 \leq X \leq 83)$.

CATEGORY 2:

7. Random variable ξ takes values in the interval (a, b), where $-\infty \leq a < b \leq \infty$, and has density $f(x)$. The function $\varphi(x)$ is strictly monotone and differentiable on (a, b). Moreover $\varphi'(x) \neq 0$ in (a, b). Find the density of the random variable $\eta = \varphi(\xi)$. Find the PDF of $\sqrt{\xi}$, if ξ has an exponential distribution with parameter λ.
8. Let X be a random variable with PDF given by:

$$f_X(x) = \begin{cases} cx^2, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   a) Find the constant c.

b) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

c) Find the CDF of r.v. X.

9. Let X be a random variable with PDF given by:

$$f_X(x) = \begin{cases} x^2\left(2x + \frac{3}{2}\right), & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X+3}$, find $\mathrm{Var}(Y)$.

10. The p.d.f. of Y is $g(y) = d \cdot y^{-4}$ for $1 < y < \infty$.

a) Find d.

b) Find the c.d.f.

c) Find $\mathbb{E}(Y)$.

d) Find m such that $\mathbb{P}(Y > m) = 0.5$.

e) Find $\mathbb{P}(Y > \mathbb{E}(Y))$.

CATEGORY 3:

11. The Probability density function (PDF) of a distribution $\mathbb{P}$, defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, is given by $p(x)$. Find the corresponding CDF for

(a) $p(x) = \frac{\theta}{\pi(\theta^2+(x-x_0)^2)}$ (Cauchy distribution with scale parameter θ and location parameter x_0);

(b) $p(x) = k(x-1)^{k-1}I(1 \leq x \leq 2)$, $k \in \mathrm{N}$.

12. A sniper in the shooting range shoots at the "quarter circle", that is, at the area $D = \{(x, y) : x^2 + y^2 < 1, x > 0, y > 0\}$. The distribution of the hitting probability $\mathbb{P}$ is uniform in the region D. In other words, the density of such a distribution is $p(x, y) = \frac{1}{\pi/4} I((x, y) \in D)$. Find

(a) The CDF and the density of the marginal probability distribution $\mathbb{P}_1$, equal to the projection of $\mathbb{P}$ along the first coordinate;

(b) The probability of the sniper hitting the square $[0, 3/4] \times [0, 3/4]$;

13. Random variable ξ has standard Cauchy distribution. Find the PDF of a) $\frac{\xi^2}{1+\xi^2}$, b) $\frac{1}{\xi}$.