Problem set 1 - Combinatorics

1. The following numbers of flights are available: Moscow-New Delhi (4), Moscow-Dubai (5), New Delhi-Sidney (3), Dubai-Sidney (6). How many ways are there to get to Sidney from Moscow.

2. There are 13 persons in a pirate ship. How many ways are there to choose a captain and his assistant?

3. There are 10 mouses in a lab. We need 5 of them to carry out an experiment. How many ways are there to choose them?

4. There are 8 different types of train cars, one of them is a restaurant car. How many different ways are there to build a train of 6 train cars, if we want the restaurant car to be present?

5. Prove the following identities:

   (a) $\binom{n}{k} = \binom{n}{n-k}$

   (b) $\overline{\binom{n}{k}} = \binom{n+k-1}{k}$

   (c) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

   (d) $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$

6. Suppose that there are 1112 cities in Russia. Aleksey and Andrey visited half of them (not together). We know that $\frac{1}{4}$ of the cities were visited for both Aleksey and Andrey. How many cities were not visited by neither of them?

7. 100 passengers are boarding a plane with 100 seats. Each passenger has a ticket with an unique seat number. How many ways are there to distribute the passengers so that none of them sit in their own seat?

8. Suppose one has 3 violets, 4 roses, and 5 tulips. How many ways are there to make a bunch of flowers so that:

   (a) all flowers are of the same kind;

   (b) the number of flowers of each kind is odd;

   (c) there are at least two flowers of different kinds?

   Two bunches are considered identical iff they are equal in number of flowers of each kind. Empty bunches are not allowed.

9. How many distinct solutions does the equation $x_1 + \ldots + x_m = n$, where $m \geq 1$ and $n \in \mathrm{N}$, have in naturals? In positive integers?

10. Count all possible six digit numbers s. t. the sum of those digits equals 29.

Extra Problems

11. Prove the following identities:

(a) $1\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = n2^{n-1}$

(b) $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$

(c) $\binom{n+m-1}{n-1} + \binom{n+m-2}{n-1} + \ldots + \binom{n-1}{n-1} = \binom{n+m}{n}$

and use (c) to prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

12. How many ways are there to distribute 15 different balls into 5 boxes in such a way that there are exactly 3 balls in each box?

13. Assume that we want to create a password using letters of the English alphabet and numbers.

    a) How many 8-digit passwords are possible?

    b) How many 8-digit passwords are possible if we are required to add at least one number?

    c) How many 8-digit passwords are possible if at least to digits are repeated?

14. Suppose that there exists an alphabet composed exclusively of the following letters: $\diamond, \square, \circ$.

    a) How many 5-letter words are there?

    b) how many words with at most 5 letters are there?

15. Two friends have a collection of different coins. One of them owns 9 coins and the other 7 coins. How many ways are there such the the two friends can exchange two coins?

16. A store sells roses in 3 different colors. How many ways are there to arrange a bouquet of 7 roses?

17. Each of some five persons has chosen a number from 1 to 100. How many ways are there for at least two of them to have chosen the same number?

18. Let $n, k \in \mathrm{N}$, s.t. $0 < k < n - 1$. Consider all possible subsets of the set n with exactly two elements. How many of those include both a number greater than k and a number lesser than k? Try to devise two different approaches to counting.

19. Suppose there are n black and n white pairwise distinct balls. Count all possible ways to choose a few balls so that there are as many whites as blacks among the chosen. A 'closed form' solution is required.

20. In a class with 30 students, 20 of them speak English, 5 - spanish, 5 - russian. Moreover, 2 - English and Russian, 2 - English and Spanish, 1 - Spanish and Russian, 1 - English, Spanish and Russian. Question: How many students do not speak any of these languages?