Problem set 10 - Expected value and Variance of Discrete Distributions

1. Let ξ_1, ξ_2 represent the values obtained in two different dice. Find $\mathsf{E}(\xi_1)$, $\mathsf{E}(\xi_1 + \xi_2)$, $\mathsf{E}(\xi_1 \xi_2)$.

2. Find the expected value and the variance of a random variable such that
   (a) $\xi \sim \mathsf{Ber}(p)$
   (b) $\xi \sim \mathsf{Bin}(n, p)$
   (c) $\xi \sim \mathsf{Pois}(\lambda)$
   (d) $\xi \sim U\{1, \dots, N\}$.
   (e) $\xi \sim Geom(p)$

3. A group of n students throw their hats at their graduation. After that, each of them picks up from the floor a hat at random. Find the expected value and the variance of the number of students who get the correct hat.

4. A box contains 100 balls, where 25 of them are white. Two balls are successively taken out of the box. Let ξ_i be the number of white balls appearing in the i-th removal ($i = 1,\ 2$). Find the correlation coefficient between ξ_1 and ξ_2.

5. A die is rolled n times. Let ξ and η correspond to the number of times we obtained 1 and 6, respectively. Find the correlation coefficient of these two random variables.

6. A coin is tossed 1000 times. Making use of the Chebyshev's Inequality, estimate the probability that the number of times we obtained Tails will be in the interval $[450, 550]$.

7. Consider the Binomial random graph $G(n, p)$. Compute the Expected value and Variance of the number of (a) isolated vertices, (b) triangles, (c) cycles of length k.

8. Let $\xi \sim \mathsf{Pois}(\lambda)$. Find $\mathsf{E}e^{\xi}$.

9. Consider the following sequence of independent random variables $\{\xi_i\}_{i=1}^n$, such that $\xi_i \sim \mathsf{Geom}(p_i)$. Prove that

$$\min\{\xi_1, \dots, \xi_n\} \sim \mathsf{Geom}\left(1 - \prod_{i=1}^{n}(1 - p_i)\right)$$

10. Two dice are rolled. Let ξ_1 correspond to the value obtained in the first die, and let ξ_2 correspond to the value obtained in the second die. On the other hand, let $\eta_1 = \xi_1 + \xi_2$ and $\eta_2 = \xi_1 - \xi_2$. Find $\mathsf{cov}(\eta_1, \eta_2)$. Are η_1 and η_2 independent?

11. A person who has n keys wants to unlock his door by testing the keys independently and in random order. We know that keys that do not match are not excluded from further tests. On the other hand, it is known that only one of the available keys fits the lock. Find the expectation and variance of the number of trials.

(a) Solve the problem provided that keys that don't fit are excluded from further tests.

Extra Problems

12. $\Omega = \{-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}\}$. The probability of each elementary outcome is $\frac{1}{5}$. Let $\xi = \cos\omega$, $\eta = \sin 2\omega$. Find $\mathsf{E}\xi$, $\mathsf{E}\eta$.

13. A box contains m white balls and n black balls. We randomly pick a ball (and then we return it) until we obtain a white ball. Find the Expected value and the Variance of the number of balls taken out.

14. A basket contains 4 apples and 6 oranges. We randomly choose from it 3 fruits. Let ξ denote the number of apples among the selected fruits. Find $\mathsf{E}\xi$, $\mathsf{Var}\xi$, $\mathsf{P}(\xi > 1)$.

15. Suppose we can roll a fair N-sided die up to m times (each face of the die contains an exclusive number of points, from 1 to N). Let ξ correspond to the number of times that face showing 1 point appears. In the same way, let η represent the sum of all the points obtained during the game.

    Find $\mathsf{E}\xi, \mathsf{Var}\xi$ and $\mathsf{E}\eta, \mathsf{Var}\eta$

16. $\Omega = \{0, \frac{\pi}{2}, \pi\}$. Let $\xi = \sin\omega$, $\eta = \cos\omega$. Find $\mathsf{cov}(\xi, \eta)$. Are ξ and η independent?

17. For a natural number k, the k-th factorial moment of a random variable ξ is $\mathsf{E}[\xi(\xi-1)(\xi-2)\ldots(\xi-k+1)]$. Prove that for $\xi \sim \mathsf{Pois}(\lambda)$, its k-th factorial moment is λ^k.