Problem set 6 - Discrete random variables. Joint distributions

1. The PMF of ξ is given by $\mathsf{P}(\xi = k) = \frac{c}{k(k+1)(k+2)}$, $k = 1, 2, \ldots$. Find **a)** The value of c. **b)** $\mathsf{P}(\xi \geq 3)$.

2. Let ξ, η be two independent random variables such that $\xi \sim Bin(n, p)$ and $\eta \sim \mathrm{Bin}(m, p)$. Find the probability distribution of $\xi + \eta$.

3. Let P be a discrete distribution on $\mathbb{R}$, $p(x) = \mathsf{P}(\{x\})$.

   (a) $p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x \in \mathbb{Z}_+$ (Poisson distribution). Find $\mathsf{P}(2\mathbb{Z}_+)$.

   (b) $p(x) = p(1-p)^{x-1}$, $x \in \mathbb{N}$ (geometric distribution). Find its CDF and $\mathsf{P}(2\mathbb{Z}_+)$.

4. A coin is tossed 3 times. Let random variable ξ denote the number of tails obtained. Let η denote the winnings earned in a single play of a game with the following rules:

   -Player wins \$1 if first tails occurs on the first toss,

   -Player wins \$2 if first tails occurs on the second toss,

   -Player wins \$3 if first tails occurs on the third toss,

   -Player loses \$1 if no tails occur.

   Find the Joint PMF of ξ and η.

5. A die is rolled k times. Let X_i, $i = 1, 2, \ldots, k$, be the number that appeared on the upper face of the die during the i-th flip.

   (a) Find the PMF of $Y_k = \max\{X_1, \ldots, X_k\}$.
   (b) Find the PMF of $Z_k = \min\{X_1, \ldots, X_k\}$.
   (c) Find the PMF of $Z_k = \max\{4, X_1\}$.

6. Let ξ and η be two random variables such that $\mathsf{P}(\xi\eta = 0) = 1; \mathsf{P}(\xi = 1) = \mathsf{P}(\xi = -1) = \mathsf{P}(\eta = 1) = \mathsf{P}(\eta = -1) = \frac{1}{4}$. Find the joint probability distribution of these two random variables and $\mathsf{E}\xi$, $\mathsf{E}\eta$, $\mathsf{Var}\xi$, $\mathsf{Var}\eta$, $\mathsf{cov}(\xi, \eta)$.

7. Let ξ, η be two independent random variables such that $\xi \sim Poiss(\lambda_1)$ and $\eta \sim \mathrm{Poiss}(\lambda_2)$. Find the probability distribution of $\xi + \eta$.

8. Let ξ, η be two independent random variables such that $\xi \sim Poiss(\lambda_1)$ and $\eta \sim \mathrm{Poiss}(\lambda_2)$. Find $\mathsf{P}\{\xi = k | \xi + \eta = n\}$.

Extra Problems

9. There are 100 cards in a box, labeled with numbers $1, 2, \ldots, 100$. A card is drawn at random from the box, and it is immediately returned to the box. We repeat this procedure 200 times. Find the approximate value of the probability that the card with the number 1 will appear exactly 3 times.

10. There is a set of four cards marked the numbers $-1, 0, 1, 2$. The player draws one card at random, writes down the number with which this card is marked, and returns the card back. Then he repeats the above procedure again. Let X_i be the number written during the i-th experiment $(i = 1, 2)$. Find the PMF of the random variables $Y = X_1 X_2$ and $Z = X_1 + X_2$.

11. The Joint PMF of the random vector (X, Y) is given by the following table:

X\Y	-1	0	1
-2	1/8	1/4	1/8
2	1/12	1/3	1/12

   (a) Find the PMF of X and $\mathsf{P}(Y \geq 0)$.

   (b) Find the PMF of $X + Y$.

   (c) Find the PMF of $Z = \min\{X, Y\}$.

12. There is a set of four cards marked the numbers $-1, 0, 1, 2$. The player draws one card at random, writes down the number with which this card is marked, and does not return the card back. Then he repeats the above procedure again. Let X_i be the number written during the i-th experiment $(i = 1, 2)$. Find the PMF of the random variables $Y = X_1 X_2$ and $Z = X_1 + X_2$.

13. You roll one red die and one green die. Define the variables ξ and η as follows:

   $\xi =$ The number showing on the red die; $\eta =$ The number of dice that show the number six. For example, if the red and green dice show the numbers 6 and 4, then $\xi = 6$ and $\eta = 1$. Write down the table showing the joint probability mass function for ξ and η, find the marginal distribution for η, and compute $\mathsf{E}\eta$.

14. The Joint PMF of the random vector (X, Y) is given by the following equality

$$\mathsf{P}(X = k, Y = j) = p^k (1 - p) \frac{\lambda^j}{j!} e^{-j}, \quad k = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots;$$

   where $0 < p < 1$, and $\lambda > 0$. Find the marginal PMF of X and Y.