Problem set 7 - Absolutely continuous Distributions

1. Let $\mathbb{P}$ be a probability distribution on $\mathbb{R}$. Let F be its CDF. Let $a < b \in \mathbb{R}$. Prove that
   (a) $\mathbb{P}([a, b]) = F(b) - F(a-)$;
   (b) $\mathbb{P}(\{a\}) = F(a) - F(a-)$.
2. The Probability density function (PDF) of a distribution $\mathbb{P}$, defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, is given by $p(x)$. Find the corresponding CDF for
   (a) $p(x) = \frac{\theta}{\pi(\theta^2+(x-x_0)^2)}$ (Cauchy distribution with scale parameter θ and location parameter x_0);
   (b) $p(x) = k(x-1)^{k-1} I(1 \le x \le 2)$, $k \in \mathrm{N}$.
3. Let $\mathbb{P}$ be a probability distribution on $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))$, defined as $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$, where $\mathbb{P}_1$ and $\mathbb{P}_2$ are uniform distributions in $[0, 1]$ and $\mathbb{P}_3$ is the exponential distribution with parameter $\lambda > 0$. Find
   (a) $\mathbb{P}\{(x, y, z) : x + z \le 3\}$;
4. A sniper in the shooting range shoots at the "quarter circle", that is, at the area $D = \{(x, y) : x^2 + y^2 < 1, x > 0, y > 0\}$. The distribution of the hitting probability $\mathbb{P}$ is uniform in the region D. In other words, the density of such a distribution is $p(x, y) = \frac{1}{\pi/4} I((x, y) \in D)$. Find
   (a) The CDF and the density of the marginal probability distribution $\mathbb{P}_1$, equal to the projection of $\mathbb{P}$ along the first coordinate;
   (b) The probability of the sniper hitting the square $[0, 3/4] \times [0, 3/4]$;
5. Let $\mathbb{P}$ be a probability measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ defined by $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$, where $\mathbb{P}_1$ — exponential distribution with parameter $\lambda_1 > 0$, $\mathbb{P}_2$ — Poisson distribution with parameter $\lambda_1 > 0$. Find $\mathbb{P}\{(x, y) : x + y \le 1\}$

   Extra Problems

6. The Probability density function (PDF) of a distribution $\mathbb{P}$, defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, is given by $p(x)$. Find the corresponding CDF for
   (a) $p(x) = xe^{-x} I(x > 0)$ (Gamma distribution with parameters 2,1);
   (b) $p(x) = \frac{1}{b-a} I(a \le x \le b)$ (uniform distribution in $[a, b]$);
7. Let $\mathbb{P}$ be a probability distribution on $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))$, defined as $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$, where $\mathbb{P}_1$ and $\mathbb{P}_2$ are uniform distributions in $[0, 1]$ and $\mathbb{P}_3$ is the exponential distribution with parameter $\lambda > 0$. Find
   (a) $\mathbb{P}\{(x, y, z) : x - y + z \ge 0\}$;