Problem set 8 - Random variables and random vectors

1. Do all properties of a probability distribution in $\mathbb{R}^2$ hold for $G_1(x, y) = I(x + y \geq 0)$?

2. Random variable ξ takes values in the interval (a, b), where $-\infty \leq a < b \leq \infty$, and has density $f(x)$. The function $\varphi(x)$ is strictly monotone and differentiable on (a, b). Moreover $\varphi'(x) \neq 0$ in (a, b). Find the density of the random variable $\eta = \varphi(\xi)$. Find the PDF of $\sqrt{\xi}$, if ξ has an exponential distribution with parameter λ.

3. Let $\xi : \Omega \to \mathbb{R}$ be a function such that $|\xi|$ is a random variable. Is ξ a random variable?

4. Let ξ, η be two random variables defined on $(\Omega, \mathcal{F})$. Let $A \in \mathcal{F}$. Prove that the following function is a random variable as well:

$$\zeta(\omega) = \xi(\omega)I(\omega \in A) + \eta(\omega)I(\omega \in \overline{A})$$

5. Let $\xi \sim \mathsf{Exp}(\lambda)$. Find the PDF of the following random variables
   (a) ξ^k, $k \in \mathrm{N}$
   (b) $\{\xi\}$, where $\{\cdot\}$ is the fractional part function
   (c) $1 - e^{-\alpha\xi}$.

6. Random variable ξ has standard Cauchy distribution. Find the PDF of a) $\frac{\xi^2}{1+\xi^2}$, b) $\frac{1}{\xi}$.

7. Let ξ have a continuous CDF F_ξ. What distribution does the random variable $F_\xi(\xi)$ have?

8. The PDF of a random vector (ξ, η) is given by $p_{(\xi,\eta)}(x, y) = \frac{1}{\pi/4} I(x^2 + y^2 < 1, x > 0, y > 0)$. Find the PDF of the random variable $\xi + \eta$.

9. Let $\xi_1, \ldots, \xi_n$ be i.i.d. random variables with CDF F_ξ. Let $\xi_{(1)}, \ldots, \xi_{(n)}$ be the same sequence but ordered in increasing order (i.e. $\xi_{(1)} \leq \ldots \leq \xi_{(n)}$). Find
   (a) the CDF of $\xi_{(k)}$
   (b) the PDF of $\xi_{(k)}$
   (c) the PDF of the random vector $(\xi_{(1)}, \xi_{(n)})$.

Extra Problems

10. Let $\xi \sim \mathsf{Exp}(\lambda)$. Find the PDF of the random variable $\frac{1}{\lambda} \ln \xi$.

11. Random variable ξ has standard Cauchy distribution. Find the PDF of
    (a) $\frac{1}{1+\xi^2}$
    (b) $\frac{2\xi}{1-\xi^2}$