

## Exam Instructions

### For CS Students

In order to pass this part of the course **Topics in Advanced Combinatorics** course, you must complete the following requirements:

- a) Successfully defend Homework 1.
- b) During the *zachet* week, you will receive three theorems chosen at random. You are required to prove at least one of them.

### For MSc Students of the “Advanced Combinatorics” Program

The final assessment for this course will be a **closed-book exam** held in January. At the exam, you will be given four theorems of varying difficulty. Each theorem will carry a maximum number of points. Your final grade (from 0 to 10) will depend on the total number of points you obtain.

However, in order to be **eligible to take the January exam**, you must successfully defend Homework 1.

## Theorems discussed in class.

**Proposition 0.1** (Lower Bound for  $R(k, k)$ ). *If*

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1,$$

*then  $R(k, k) > n$ . Thus  $R(k, k) > \lfloor 2^{k/2} \rfloor$  for all  $k \geq 3$ .*

**Theorem 0.2** (Property  $S_k$ ). *We say that  $T$  has the property  $S_k$  if, for every set of  $k$  players, there exists one that beats them all. If*

$$\binom{n}{k} (1 - 2^{-k})^{n-k} < 1,$$

*then there exists a tournament on  $n$  vertices that has the property  $S_k$ .*

**Definition 0.3.** A *dominating set* of an undirected graph  $G = (V, E)$  is a set  $U \subseteq V$  such that every vertex  $v \in V - U$  has at least one neighbor in  $U$ .

**Theorem 0.4.** *Let  $G = (V, E)$  be a graph on  $n$  vertices, with minimum degree  $\delta > 1$ . Then  $G$  has a dominating set of at most*

$$n \cdot \frac{1 + \ln(\delta + 1)}{\delta + 1}$$

*vertices.*

Let  $H = (V, E)$  be an  $n$ -uniform hypergraph, meaning that each of its edges contains precisely  $n$  vertices. We say that  $H$  has *property  $B$* , or that it is *two-colorable*, if there exists a two-coloring of  $V$  such that no edge is monochromatic. Let  $m(n)$  denote the minimum possible number of edges of an  $n$ -uniform hypergraph that does not have property  $B$ .

**Proposition 0.5** (Erdős (1963a)). *Every  $n$ -uniform hypergraph with fewer than  $2^{n-1}$  edges has property B. Therefore,  $m(n) \geq 2^{n-1}$ .*

**Theorem 0.6.**

$$m(n) < (1 + o(1)) \frac{e \ln 2}{4} n^2 2^n.$$

**Definition 0.7.** A family  $\mathcal{F}$  of sets is called *intersecting* if  $A, B \in \mathcal{F}$  implies  $A \cap B \neq \emptyset$ .

Suppose  $n \geq 2k$ , and let  $\mathcal{F}$  be an intersecting family of  $k$ -element subsets of an  $n$ -set, for definiteness  $\{0, \dots, n-1\}$ .

**Theorem 0.8** (Erdős–Ko–Rado).  $|\mathcal{F}| \leq \binom{n-1}{k-1}$ .

**Theorem 0.9** (Erdős, 1959). *For all integers  $k, \ell$ , there exists a graph  $G$  with*

$$\text{girth}(G) > \ell \quad \text{and} \quad \chi(G) > k.$$

**Theorem 0.10.** *There exists a tournament  $T$  with  $n$  players and at least*

$$n! 2^{-(n-1)}$$

*Hamiltonian paths.*

**Theorem 0.11.** *Let  $G = (V, E)$  be a graph with  $n$  vertices and  $e$  edges. Then  $G$  contains a bipartite subgraph with at least  $e/2$  edges.*

**Theorem 0.12.** *If  $G$  has  $2n$  vertices and  $e$  edges, then it contains a bipartite subgraph with at least  $en/(2n-1)$  edges. If  $G$  has  $2n+1$  vertices and  $e$  edges, then it contains a bipartite subgraph with at least*

$$e(n+1)/(2n+1)$$

*edges.*

**Theorem 0.13.** *Let  $G = (V, E)$  have  $n$  vertices and  $nd/2$  edges, where  $d \geq 1$ . Then*

$$\alpha(G) \geq \frac{n}{2d}.$$

**Theorem 0.14** (Erdős–Szekeres, 1935). *Let  $R(s, t)$  denote the Ramsey number for integers  $s, t \geq 2$ . Then*

$$R(s, t) \leq R(s-1, t) + R(s, t-1).$$

**Corollary 0.15.** *For all integers  $s, t \geq 2$ , the Ramsey number satisfies*

$$R(s, t) \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}.$$

**Corollary 0.16.** *For all integers  $s \geq 2$ , we have*

$$R(s, s) \leq \binom{2s-2}{s-1}.$$

**Theorem 0.17** (Erdős). *For every integer  $s \geq 2$ ,*

$$R(s, s) \geq (1 + o(1)) \frac{s}{e\sqrt{2}} 2^{s/2} \quad \text{as } s \rightarrow \infty.$$

**Theorem 0.18** (Lovász Local Lemma, symmetric form). Let  $A_1, \dots, A_n$  be events in a probability space. Assume that each event  $A_i$  satisfies

$$\mathbb{P}(A_i) \leq p$$

and that each  $A_i$  is independent of all but at most  $d$  other events. If

$$ep(d+1) \leq 1,$$

then

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{A_i}\right) > 0.$$

**Theorem 0.19** (Lovász Local Lemma, general (asymmetric) form). Let  $A_1, \dots, A_n$  be events in a probability space. Suppose there exist real numbers  $x_1, \dots, x_n \in (0, 1)$  such that for every  $i$ ,

$$\mathbb{P}(A_i) \leq x_i \prod_{A_j \sim A_i} (1 - x_j),$$

where the product is taken over all events  $A_j$  that are dependent on  $A_i$ . Then

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{A_i}\right) \geq \prod_{i=1}^n (1 - x_i) > 0.$$

**Theorem 0.20.** Let  $s \geq 2$  be an integer. If there exists an integer  $n$  such that

$$e \binom{s}{2} \binom{n-2}{s-2} 2^{1-\binom{s}{2}} \leq 1,$$

then

$$R(s, s) > n.$$

**Corollary 0.21.** As  $s \rightarrow \infty$ , the diagonal Ramsey numbers satisfy

$$R(s, s) \geq (1 + o(1)) \frac{\sqrt{2}}{e} s 2^{s/2}.$$

**Theorem 0.22** (Erdős–Rényi). As  $t \rightarrow \infty$ , the off-diagonal Ramsey numbers satisfy

$$R(3, t) \geq \left(\frac{1}{4} + o(1)\right) \frac{t^2}{\ln t}.$$

**Theorem 0.23** (Erdős–Lovász). Let  $H$  be an  $n$ -uniform hypergraph, where  $n \geq 3$ . If the degree of every vertex of  $H$  is at most  $n$ , then

$$\chi(H) \leq 2.$$

**Theorem 0.24.** Let  $p = \frac{1}{2}$ . Then for every  $\varepsilon > 0$ , with high probability,

$$\frac{\chi_{gr}(G)}{\chi(G)} \leq 2 + \varepsilon, \quad \frac{\alpha(G)}{\alpha_{gr}(G)} \leq 2 + \varepsilon.$$

Within this course, only the inequality concerning the independence number will be proved. The proof for the chromatic number proceeds analogously.

## 0.1 Theorems discussed before October 1

**Theorem 0.25.** Let  $p = p(n) = \frac{c \ln n}{n}$ , where  $c$  is a constant and  $c > 0$ .

1. If  $c > 1$ , then with high probability the random graph  $G(n, p)$  is connected.
2. If  $c < 1$ , then with high probability the random graph  $G(n, p)$  is disconnected.