List of Theorems for Exam in Graph Theory

Theorem 1 (Erdős–Ko–Rado)**.** *Let $n \geq 2k$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family, i.e., for all $A, B \in \mathcal{F}$ we have $A \cap B \neq \emptyset$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Moreover, if $n > 2k$, equality holds if and only if $\mathcal{F}$ consists of all k-subsets containing a fixed element.

Theorem 2 (Lovász, Chromatic Number of Kneser Graphs)**.** *Let $KG(n,k)$ be the Kneser graph whose vertices are the k-subsets of $[n]$ and edges connect disjoint sets. Then*

$$\chi(KG(n,k)) = n - 2k + 2.$$

Theorem 3 (Borsuk–Ulam)**.** *Let $A_1, \dots, A_{n+1}$ be closed (or open) subsets of the sphere S^n such that*

$$S^n = A_1 \cup \cdots \cup A_{n+1}.$$

Then there exists an index i and a point $x \in S^n$ such that $x \in A_i \quad \text{and} \quad -x \in A_i$.

Definition 1. *Let n, k, t be positive integers. Define $m(n,k,t)$ to be the maximum size of a family $\mathcal{F} \subseteq \binom{[n]}{k}$ such that for all distinct sets $A, B \in \mathcal{F}$ we have $|A \cap B| \neq t$. In other words,*

$$m(n,k,t) = \max\left\{ |\mathcal{F}| : \mathcal{F} \subseteq \binom{[n]}{k},\ \forall A \neq B \in \mathcal{F},\ |A \cap B| \neq t \right\}.$$

Theorem 4. *For $m(n,3,1)$ we have*

$$m(n,3,1) \geq \begin{cases} n, & \text{if } 4|n, \\ n-1, & \text{if } n \equiv 1 \pmod 4, \\ n-2, & \text{otherwise.} \end{cases}$$

Theorem 5. *For $m(n,5,2)$ we have*

$$m(n,5,2) \leq \binom{n}{2} + 2\binom{n}{1} + \binom{n}{0}.$$

Theorem 6. *The previous bound can be improved as follows:*

$$m(n,5,2) \leq \binom{n}{2}.$$

Theorem 7. $m(n,5,2) \geq \binom{n-3}{2}$.

Theorem 8 (Frankl–Wilson, 1981)**.** *Let n, k, t be integers such that $k - t = p$, where p is a prime, and $k < 2p$. Then*

$$m(n,k,t) \leq \sum_{i=0}^{p-1} \binom{n}{i}.$$

Definition 2. *Let $H = (V, E)$ be a hypergraph. A set $T \subseteq V$ is called a vertex cover if*

$$T \cap e \neq \emptyset \quad \text{for all } e \in E.$$

The vertex cover number $\tau(H)$ is defined as

$$\tau(H) = \min\{\, |T| : T \subseteq V,\ T \cap e \neq \emptyset\ \forall e \in E \,\}.$$

Theorem 9. *For all n, k, s and any k-uniform hypergraph H on n vertices with $|E(H)| = s$, we have*

$$\tau(H) \le \max\left\{\frac{n}{k}, \frac{n}{k}\ln\frac{sk}{n}\right\} + \frac{n}{k} + 1.$$

Theorem 10. *Let $n \ge 4$, $k \le \frac{n}{4}$, and s satisfy $4 \le \frac{sk}{n} \le k$.*

Then there exists a hypergraph H with $|E(H)| = s$ such that

$$\tau(H) \ge \frac{1}{32} \cdot \frac{n}{k} \ln\frac{sk}{n}.$$

Theorem 11 (Max-Flow Min-Cut)**.** *Let $G = (V, E)$ be a directed network with capacity function $c : E \to \mathbb{R}_{\ge 0}$, source $s \in V$, and sink $t \in V$. Then the maximum value of an s–t flow is equal to the minimum capacity of an s–t cut, i.e.,*

$$\max_f |f| = \min_{(S,T)} c(S, T),$$

where the minimum is taken over all partitions (S, T) of V such that $s \in S$ and $t \in T$, and

$$c(S, T) = \sum_{u \in S,\, v \in T} c(u, v).$$

Theorem 12 (Hall's Marriage Theorem)**.** *Let $G = (X \cup Y, E)$ be a bipartite graph. Then there exists a matching that covers all vertices of X if and only if for every subset $S \subseteq X$ we have*

$$|N(S)| \ge |S|,$$

where $N(S)$ denotes the set of neighbors of S in Y.

Theorem 13 (Menger's Theorem, vertex version)**.** *Let $G = (V, E)$ be a finite graph and let $x, y \in V$, $x \neq y$. Then the maximum number of internally vertex-disjoint x–y paths is equal to the minimum size of a vertex set $S \subseteq V \setminus \{x, y\}$ whose removal separates x and y.*

Theorem 14 (Menger's Theorem, edge version)**.** *Let $G = (V, E)$ be a finite graph and let $x, y \in V$, $x \neq y$. Then the maximum number of edge-disjoint x–y paths is equal to the minimum number of edges whose removal separates x and y.*

Theorem 15 (Brooks' Theorem)**.** *Let G be a connected graph with maximum degree Δ.*

If G is neither a complete graph nor an odd cycle, then $\chi(G) \le \Delta$.

Theorem 16 (Kőnig's Theorem)**.** *For every bipartite graph G,*

$\chi'(G) = \Delta(G)$, where $\chi'(G)$ is the edge chromatic number of G and $\Delta(G)$ is the maximum degree of G.

Theorem 17 (Vizing's Theorem)**.** *For every simple graph G, $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$, where $\chi'(G)$ is the edge chromatic number of G.*

Theorem 18 (Turán, 1941)**.** *Let $|G| = n, \omega(G) \le k$, and suppose that G has the largest possible number of edges among all graphs satisfying these conditions.*

Then G is a complete k-partite graph in which the sizes of any two parts differ by at most 1.